Lecture 25.

Moser Normal form

$M \subseteq \mathbb{C}^{n+1}$ (s.ψcvx, C^ω), $p_0 = (0,0)$

$$\operatorname{Im} w = |z|^2 + N(z,\bar z, \operatorname{Re} w), \quad \text{where}$$

$$N(z,\bar z, s) = \sum_{\underline{\underline{k,l \ge 2}}} N_{kl}(z,\bar z, s)$$

polynomial in $z,\bar z$ of bideg $=(k,l)$

and

$$\text{(Trace)} \quad \operatorname{tr} N_{22} = \operatorname{tr}^2 N_{23} = \operatorname{tr}^3 N_{33} = 0.$$

Moreover, given two normal forms $((z,w),N)$, $((z',w'),N')$ $\exists$ unique $\Phi \in \operatorname{Aut}_0(M_0)$ and $H : (\mathbb{C}^{n+1},0) \circlearrowleft$ bihol. s.t. $(z',w') = (H \circ \Phi)(z,w)$.

Recall. • $\operatorname{tr} P(z,\bar z, s) = \Delta_z P(z,\bar z, s) = 4 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar z_j} P$.

• One easily sees that if $n=1$ ($M \subseteq \mathbb{C}^2$), then (Trace) $\Leftrightarrow N_{22} = N_{23} = N_{33} = 0$.

• Lowest degree term that appears in $\mathbb{C}^2$ is N_{24}. E. Cartan's 6th order tensor

CR curvature

Suppose $n \geq 2$. Then lowest order term is N_{22} and we set $S(z,\bar z) = N_{22}(z,\bar z,0)$. Then S is a polynomial of bideg $=(2,2)$ and $\Delta S = 0$ (S is <u>harmonic</u>).

We can write

$$S(z,\bar z) = \sum_{\alpha,\beta,\nu,\mu=1}^{n} S_{\alpha\bar\beta\nu\bar\mu}\, z_\alpha \overline{z_\beta}\, z_\nu \overline{z_\mu}$$

in such a way that $S_{\alpha\bar\beta\nu\bar\mu}$ has <u>Hermitian curvature symmetries</u>

$$S_{\alpha\bar\beta\nu\bar\mu} = S_{\nu\bar\beta\alpha\bar\mu} = S_{\alpha\bar\mu\nu\bar\beta}$$

$$S_{\alpha\bar\beta\nu\bar\mu} = \overline{S_{\beta\bar\alpha\mu\bar\nu}}$$

<u>Def</u> $S_{\alpha\bar\beta\nu\bar\mu}$ is the CR curvature tensor of M at p_0. $S(z,z)$ <u>sectional CR curv.</u>

We check that $S_{\alpha\bar\beta\nu\bar\mu}$ transforms as a tensor.

- A calculation show that $G_0 = \mathrm{Aut}_0(M_0)$ (isotropy subgroup) is given by

$$\Phi : \begin{cases} z' = \dfrac{U(z+aw)}{1+b\cdot v+cw} \\ w' = \dfrac{rw}{1+b\cdot v+cw}, \end{cases}$$

where $a, b \in \mathbb{C}^n$, $c \in \mathbb{C}$, $r > 0$ and

$$U^* U = rI.$$

- In the proof of Moser's thm, one notes that the bihol. $H : (\mathbb{C}^{n+1}, 0) \circlearrowleft$ satisfies $H = (f, g)$, $f = (f_1, \dots, f_n)$

$$\begin{cases} f(z,w) = z + O(|(z,w)|^2) \\ g(z,w) = w + O(|(z,w)|^2) \end{cases}$$

$$\left(+ \frac{\partial^2 g}{\partial z_i \partial z_j}(0) = \mathrm{Re}\, \frac{\partial^2 g}{\partial w^2}(0) = 0 \right)$$

One may now check how the secto. curv. $S(z,\bar{z})$ transforms under $(z',w') = (H\circ\Phi)(z,w)$ in terms of the first order derivatives of Φ: $\underline{U, a, r}$.

Let $S'_{\alpha\bar\beta\gamma\bar\mu}$, $S_{\alpha\bar\beta\gamma\bar\mu}$ denote the CR curvature terms in the normal forms $((z',w'), N')$, $((z,w), N)$ respectively.

$$S_{\alpha\bar\beta\nu\bar\mu} = r^{-1} S'_{\gamma\bar\varepsilon\delta\bar\kappa}\, U_\alpha^\gamma\, \overline{U_\beta^\varepsilon}\, U_\nu^\delta\, \overline{U_\mu^\kappa}$$

That is, $S_{\alpha\bar\beta\nu\bar\mu}$ transforms as a tensor of a (suitable) tensor bundle.

<u>Def.</u> M is <u>umbilical</u> at p_0 if $S_{\alpha\bar\beta\nu\bar\mu} = 0$ (i.e. in some Moser normal form and therefore in every normal form).

Another way to describe umbilicity is to note that M_0 osculates M to 4:th order in general but to 5:th order precisely when $S_{\alpha\bar\beta\nu\bar\mu} = 0 \iff p_0$ umbilical point.

Thm (Webster, Cartan, Chern)

If $S_{\alpha\bar\beta\gamma\bar\mu} = 0$ for all $p \in M$ near p_0 (i.e. M is umbilical $\forall p$ near p_0), then M is spherical at p_0 (i.e. $\exists$ local CR diffeomorphism $h : (M, p_0) \to M_0 \cong S^{2n+1}$).

To prove this:

$\mathrm{Aut}_0(M_0)$
$\wr$

(1) (Chern, Cartan). Set up a G_0-structure

$$\begin{array}{c} G_0 \circlearrowright P \cong SU(1, n+1) \\ \downarrow \\ M \end{array}$$

and find parallelism (unique coframe)

(2) (Webster, Chern) Show that invariants in Cartan's method of equivalence are all covariant derivatives of CR curvature.

(3) Cartan's method of equivalence $\Rightarrow$ Thm.

No time for all this background (~)